MATH4010 Functional Analysis

Homework 5 suggested Solution

Question 1. Let (x_n) and (y_n) be the sequences in a Hilbert space H. Suppose that the limits $\lim \|x_n\|$, $\lim \|y_n\|$ and $\lim \left\| \frac{x_n+y_n}{2} \right\|$ exist and are equal. Show that if (x_n) is convergent, then so is (y_n).

Solution: By the parallelogram law,

$$\|x_n - y_n\|^2 = 2\|x_n\|^2 + 2\|y_n\|^2 - \|x_n + y_n\|^2 .$$

Note that

$$\begin{aligned}
&\lim(2\|x_n\|^2 + 2\|y_n\|^2 - \|x_n + y_n\|^2) \\
&= 2(\lim \|x_n\|)^2 + 2(\lim \|y_n\|)^2 - 4\left(\lim \left\| \frac{x_n + y_n}{2} \right\|\right)^2 \\
&= 0
\end{aligned}$$

The last equality follows by assumption. Thus, $\lim \|x_n - y_n\|^2 = 0$, i.e. $(x_n - y_n)$ converges to 0. Since (x_n) is convergent, so is (y_n).

Question 2. Fix an element $z \in H$. Define a linear functional φ on H by $\varphi(x) = (x, z)$.

(i) Show that $\|\varphi\| = \|z\|$.

(ii) Let $w \in H$. Find $dist(w, \ker \varphi)$, the distance between the element w and $\ker \varphi$. (the answer is in terms of w and z.)

(iii) Let $H = L^2(\mathbb{T})$ and φ be the functional on H given by $\varphi(f) := \int_{\mathbb{T}} f(z)dz$ for $f \in H$. Let $g \in H$. Find the element $h \in \ker \varphi$ such that $\|g - h\| = dist(\mathrm{g}, \ker\varphi)$.

Solution:

(i) By the Cauchy-Schwarz Inequality,

$$|\varphi(x)| = |(x, z)| \leq \|x\|\|z\|. \tag{1}$$

Hence φ is bounded with $\|\varphi\| \leq \|z\|$. Equality (1) is achieved when $x = z$. Therefore, $\|\varphi\| = \|z\|$.

(ii) If $z = 0$, then kerφ = H. Therefore, $dist(w, \ker\varphi) = 0$. We suppose that $z \neq 0$ in the following. For each $x \in \ker\varphi$, $|(w, z)| = |(w - x, z) \leq \|w - x\|\|z\|$. Thus $\|w - x\| \geq \frac{|(w,z)|}{\|z\|}$, which implies that $dist(w, \ker\varphi) \geq \frac{|(w,z)|}{\|z\|}$.

Take $D = \frac{(w,z)}{\|z\|^2}$, and denote $y = w - Dz$, we have

$$\varphi(y) = (w, z) - D\|z\|^2 = 0.$$

So $y \in \ker\varphi$ with $\|w - y\| = \|Dz\| = \frac{|(w,z)|}{\|z\|}$. Therefore $dist(w, \ker\varphi) = \frac{|(w,z)|}{\|z\|}$.

(iii) Let $g_0 \equiv 1$ be the constant 1 function. Then we note that

$$(g, g_0) = \int_{\mathbb{T}} g(z) \cdot 1dz = \varphi(g), \quad \text{for all g} \in \text{H}.$$

Following from 2(ii), there exists $h \in \ker\varphi$ given by

$$h = g - \frac{(g, g_0)}{\|g_0\|^2} g_0 = g - \int_{\mathbb{T}} f(z)dz,$$

such that $\|g - h\| = dist(\text{g}, \ker\varphi)$.